$X = \text{var.} \Rightarrow k(X)$ isomorphic invariant

How to describe the classification with this invariant?

$\Downarrow$

$\text{tr.deg}\, k(X)$ iso. invar.

classification all var. with dim.

$U \hookrightarrow X$ & $V \hookrightarrow Y$ & $U \cong V \Rightarrow k(X) \cong k(Y)$

Lemma: Let X, Y be two var. with isomorphic rational function fields. Then there exists $U \hookrightarrow X$ & $V \hookrightarrow Y$ s.t. $U \cong V$.

Pf: WLOG. WMA: $X \hookrightarrow \mathbb{A}^n$, $Y \hookrightarrow \mathbb{A}^m$

$$\begin{cases} k(X) \cong k(x_1, \dots, x_n) & x_i = X_i \bmod I(X) \\ k(Y) \cong k(y_1, \dots, y_m) & y_i = Y_i \bmod I(Y) \end{cases}$$

$$k(X) \xrightarrow[\cong]{\varphi} k(Y)$$

$$\left.\begin{array}{ccc} x_i & \longmapsto & \dfrac{f_i(y)}{g_i(y)} \\ \dfrac{l_i(x)}{h_i(x)} & \longleftarrow & y_i \end{array}\right\} \Rightarrow \begin{cases} g := g_1 \cdots g_n \in \Gamma(Y) \subseteq k(Y) \\ h := h_1 \cdots h_m \in \Gamma(X) \subseteq k(X) \end{cases}$$

$$\Rightarrow \begin{cases} \varphi^{-1}(g) \subseteq \varphi^{-1}(\Gamma(Y)) \subseteq k[x_1, \dots, x_n][\frac{1}{h}] \\ \varphi(h) \subseteq \varphi(\Gamma(X)) \subseteq k[y_1, \dots, y_n][\frac{1}{g}] \end{cases}$$

$$k[x_1, \dots, x_n]\left[\frac{1}{h\, \varphi^{-1}(g)}\right] \xrightarrow[\cong]{\varphi} k[y_1, \dots, y_n]\left[\frac{1}{g\, \varphi(h)}\right]$$

$$\Rightarrow X_{h\varphi^{-1}(g)} \xrightarrow{\cong} Y_{g\varphi(h)} \Rightarrow \checkmark$$

to find if $K(X) \cong k(Y)$, we only need to find an local iso.

$X \hookleftarrow U \xrightarrow{\cong} V \hookrightarrow Y$. $\longleftarrow$ morphism defined on open subvar.

§6.6. Rational Maps

X, Y = varieties.

$$\begin{tikzcd}
 & U_1 \arrow[dr, hook] \arrow[drr, bend left, "f_1"] & & \\
U_1 \cap U_2 \arrow[ur, hook] \arrow[dr, hook] & & X \arrow[r, dashed] & Y \\
 & U_2 \arrow[ur, hook] \arrow[urr, bend right, "f_2"'] & &
\end{tikzcd}$$

f_1 & f_2 are equivalent $\overset{\text{def}}{\Longleftrightarrow}$ $f_1|_{U_1\cap U_2} = f_2|_{U_1\cap U_2}$ 延拓存在则唯一

注: Cor to Prop 7 in §6.4 $\Rightarrow$ f_i is uniquely determined by $f_i|_{U_1\cap U_2}$

An equivalence class is called a rational map from X to Y. (Simply denote $f: X \dashrightarrow Y$.)

domain (U) of a rational map $:= \bigcup_{f_\alpha} U_\alpha$ $\quad (f_\alpha : U_\alpha \to Y)$

$f: U \to Y$ s.t. $f|_{U_\alpha} = f_\alpha$

Fact: 0) $\forall p \in U$, $f(p) \in Y$ is well-defined.

1) f is a morphism (to be a morphism is a local property)

2) every equivalent morphism is a restriction of f.

$V := \overline{f(U)} \hookrightarrow Y$ closure of $f(U)$ in Y

(U ← domain.)

Lemma: $V_\alpha :=$ closure of $f_\alpha(U_\alpha)$ in Y. Then $V_\alpha = V$.

(14) Pf: $f_\alpha(U_\alpha) \subseteq f(U) \Rightarrow V_\alpha \hookrightarrow V \Rightarrow U_\alpha \subseteq f^{-1}(V_\alpha) \hookrightarrow f^{-1}(V) = U \Rightarrow f^{-1}(V_\alpha) = U \Rightarrow V$

Lemma: X, Y = affine var. $\quad f: X \to Y$ morphism.

$f(X)$ is dense in Y $\Leftrightarrow$ $\tilde{f}: \Gamma(Y) \to \Gamma(X)$ is injective.

Pf: $\Rightarrow$): $\forall g \in \Gamma(Y)$, if $\tilde{f}(g) = 0 \Rightarrow g \circ f\big|_X = 0 \Rightarrow g\big|_{f(X)} = 0 \overset{f(X) \text{ dense in } Y}{\Longrightarrow} g\big|_Y = 0 \Rightarrow g = 0$

$\Leftarrow$): Suppose not. $V := \overline{f(X)} \subsetneq Y$. $\forall H \in I(V) \setminus I(Y)$

$\Rightarrow h := H \bmod I(Y) \in \Gamma(Y) \setminus \{0\}$ & $h\big|_V = 0$

$\Rightarrow \tilde{f}(h)\big|_X = (h\big|_V \circ f)\big|_X = 0 \Rightarrow \tilde{f}(h) = 0 \Rightarrow \tilde{f} \neq$ inj. ↯.

to study $f: X \dashrightarrow Y$, we only need to consider $f: X \dashrightarrow V$, i.e.

we may assume: $f(U)$ is dense in Y.

Def: 1) $f: X \dashrightarrow Y$ is called **dominating** if $f_\alpha(U_\alpha) \subset Y$ dense. for some α

(independent on α)

Prop 11. (1). $F: X \dashrightarrow Y$ dominating. $\Rightarrow \Gamma(V) \overset{\tilde{f}}{\hookrightarrow} \Gamma(U)$.

$$\begin{tikzcd} X \arrow[r, dashed, "F"] & Y \\ U \arrow[u, hook, "\text{affine}"] \arrow[r, "f"] & V \arrow[u, hook', "\text{affine}"'] \end{tikzcd}$$

$\Rightarrow \tilde{F}: k(Y) \hookrightarrow k(X)$

indep. on choice of f

(2). $\mathrm{Hom}_{k\text{-alg}}(k(Y), k(X)) \overset{1:1}{\longleftrightarrow} \{F: X \dashrightarrow Y \mid \text{dominating}\}$

Pf (1) Prob 6.26

(2). assume X, Y = affine.

$\forall \varphi: k(Y) \to k(X) \Rightarrow \varphi(\Gamma(Y)) \subset \Gamma(X_b)$ for some b

$\Rightarrow f: X_b \overset{\text{dense}}{\to} Y$.

Def: $(A, m_A), (B, m_B)$ = loc. ring. $A \subset B$. B **dominates** A if $m_A \subseteq m_B$.

Lemma: Let $F: X \dashrightarrow Y$ be a dominating rational map. $\forall P \in X, Q \in Y$.

$$\left.\begin{array}{l} P \in \underbrace{U(F)}_{\text{domain of } F} \\ Q = F(P) \end{array}\right\} \Leftrightarrow \mathcal{O}_P(X) \text{ dominates } \tilde{F}(\mathcal{O}_Q(Y))$$

Pf: $\Rightarrow$): clear

$\Leftarrow$): $P \in V$, $Q \in W$ affine neighborhood

assume $\Gamma(W) = k[y_1, \cdots, y_n]$. $\tilde{F}(y_i) = \frac{a_i}{b_i}$ $\left(\begin{array}{l} a_i, b_i \in \Gamma(V) \\ b_i(P) \neq 0 \end{array}\right)$

$b = b_1 \cdots b_n \Rightarrow \tilde{F}(\Gamma(W)) \subset \Gamma(V_b)$

$\Rightarrow \exists!\ f: V_b \to W$

$\forall g \in \Gamma(W)$, $g(Q) = 0 \Rightarrow g \in m_Q \Rightarrow \tilde{F}(g) \in m_P$

$\Rightarrow g \circ f(P) = \tilde{F}(g)(P) = 0$

$\Rightarrow f(P) = Q$

Def (1) $F: X \dashrightarrow Y$ **birational** if $\exists\, U \overset{\neq \phi}{\hookrightarrow} X$, $V \overset{\neq \phi}{\hookrightarrow} Y$ s.t. $F: U \xrightarrow{\cong} V$

(2) X & Y are **birational equivalent** $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ birational $X \dashrightarrow Y$.

Fact: 1) $U \overset{\neq \phi}{\hookrightarrow} X$ var $\Rightarrow$ U & X b.equ.

2) $\mathbb{A}^n$ & $\mathbb{P}^n$ b.equ.

3) X & Y : b.equ $\Leftrightarrow$ $k(X) \cong k(Y)$.

Def: A var. is called **rational** if it is b.equ. to $\mathbb{A}^n$ (or $\mathbb{P}^n$).

Thm: Let X be a variety of dim r. Then X is birational to an closed subvariety of $\mathbb{A}^{r+1}$.

Cor: Every curve is birationally equivalent to a plane curve.

Lemma (a): Let L be a finite separable extension of K, then $\exists z \in L$ s.t. $L = K(z)$.

Lemma (b): Let K is a f.g. field over k with $\mathrm{tr.deg}_k K = r$. Then $K = k(x_1, \cdots, x_r, x_{r+1})$ for some $x_i \in K$.

Pf of thm: $X = $ var. of dim $r \overset{\text{Lem b}}{\Rightarrow} k(X) = k(x_1, \cdots, x_{r+1})$ for some $x_i \in k(X)$

$I := \ker\left(k[X_1, \cdots, X_{r+1}] \twoheadrightarrow k[x_1, \cdots, x_{r+1}] \subseteq k(x_1, \cdots, x_{r+1})\right)$

prime

$\Rightarrow V' = V(I) \subseteq \mathbb{A}^{r+1}$ var. with.

$\Gamma(V') = k[X_1, \cdots, X_{r+1}]/I \cong k[x_1, \cdots, x_{r+1}] \Rightarrow k(V') = k(X)$

$\Rightarrow X$ is birational to V'. $\square$